Concrete algorithms for word problem and subsemigroup problem for semigroups which are disjoint unions of finitely many copies of the free monogenic semigroup

Nabilah Abughazalah*
Department of Mathematical Sciences
Princess Nourah bint Abdulrahman University
Riyadh, Saudi Arabia
E-mail: nhabughazala@pnu.edu.sa

December 8, 2016

Abstract

Every semigroup which is a finite disjoint union of copies of the free monogenic semigroup (natural numbers under addition) has soluble word problem and soluble membership problem. Efficient algorithms are given for both problems.

1 Introduction

It is well known that some semigroups may be decomposed into a disjoint union of subsemigroups which is unlike the structures of classical algebra such as groups and rings. For instance, the Rees Theorem states that every completely simple semigroup is a Rees matrix semigroup over a group G, and is thus a disjoint union of copies of G, see [7, Theorem 4.2.1]; every Clifford semigroup is a strong semilattice of groups and as such it is a disjoint union of its maximal subgroups, see

*The author is financially supported by Princess Nourah bint Abdulrahman University in Riyadh and Saudi Aramco Ibn Khaldun Fellowship for Saudi Women, in partnership with the Center for Clean Water and Clean Energy at MIT.
Key words and phrases: Semigroup, Decidability, Word problem, Membership problem.
Mathematics Subject Classification: 20M05.

[7, Theorem IV.2.2]; every commutative semigroup is a semilattice of archimedean semigroups, see [5, Theorem 3.3.1].

If S is a semigroup which can be decomposed into a disjoint union of subsemigroups, then it is natural to ask how the properties of S depend on these subsemigroups. For example, if the subsemigroups are finitely generated, then so is S. Araújo et al.[3] consider the finite presentability of semigroups which are disjoint unions of finitely presented subsemigroups; Golubov [4] showed that a semigroup which is a disjoint union of residually finite subsemigroups is residually finite.

In the context where S is a semigroup which is a disjoint union of finitely many copies of the free monogenic semigroup, the authors in [1] proved that S is finitely presented and residually finite; in [2] the authors proved that, up to isomorphism and anti-isomorphism, there are only two types of semigroups which are unions of two copies of the free monogenic semigroup. Similarly, they showed that there are only nine types of semigroups which are unions of three copies of the free monogenic semigroup and provided finite presentations for semigroups of each of these types.

In this paper we continue investigating finiteness conditions for a semigroup which is a disjoint union of finitely many copies of the free monogenic semigroup, the decidability of the word problem and membership problem in particular.

The paper is organized as follows. In section 2 we recall some lemmas from [1] and explain the obtained results with clarify the strong regularities which are all described in terms of arithmetic progressions. In Section 3 we prove that S has a soluble word problem and soluble membership problem.

2 Properties of the semigroup which is a disjoint union of finitely many copies of the free monogenic semigroup

Let S be a semigroup which is a disjoint union of finitely many copies of the free monogenic semigroup:

$$S = \bigcup_{a \in A} N_a,$$

where A is a finite set and $N_a = \langle a \rangle$ for $a \in A$. We proved in [1, Theorem 3.1] that the semigroup S has the finite presentation

$$\big\langle A \mid a^k b = [\alpha(a,k,b,1)]^{\kappa(a,k,b,1)},\ (a,b \in A,\ k \in \{1,2,\ldots,j\})\big\rangle, \tag{1}$$

for some $\alpha(a,k,b,1) \in A$ and $\kappa(a,k,b,1) \in \mathbb{N}$.

We introduce the necessary lemmas from the paper [1] to add more information to the presentation (1).

Lemma 2.1 ([1], Lemma 2.4). *If*

$$a^p x = b^r,\ a^{p+q} x = b^{r+s}$$

for some $a, b \in A$, $x \in S$, $p, q, r \in \mathbb{N}$, $s \in \mathbb{N}_0$, *then*

$$a^{p+qt}x = b^{r+st}$$

for all $t \in \mathbb{N}_0$.

Lemma 2.2. *Let* $a, c \in A, b \in S$. *If* $a^p b = c^{n_p}$ *for two distinct values of* p *then there exists an arithmetic progression* $p + qn$, $r \in \mathbb{N}$, $s \in \mathbb{N} \cup \{0\}$ *such that* $a^{p+qn}b = c^{r+sn}$ *for every* $n \in \{0, 1, 2, ...\}$.

Proof. Since, $a^p b = c^{n_p}$ for two distinct values of p, then we have q, r, such that $a^q b = c^{n_q}$, $a^r b = c^{n_r}$ where $q \leq r$ and $r - q$ is as small as possible. Hence, by Lemma 2.1, $a^{r+n(r-q)}b = c^{n_r+n(n_r-n_q)}$ holds for every $n \in \mathbb{N}$. □

Lemma 2.3. *Let* $a, c \in A, b \in S$. *Suppose* $q \in \mathbb{N}$ *is the smallest possible number such that* $a^p b = c^r$, $a^{p+q}b = c^{r+s}$ *for some* $p, r \in \mathbb{N}, s \in \mathbb{N} \cup \{0\}$ *holds in* S. *Then if* $i \in \{p+1, p+2, ..., p+q-1\}$ *and* $a^i b = d^t$ *we have* $d \neq c$.

Proof. Suppose to the contrary that $d = c$ then we have

(i) $a^p b = c^r$;

(ii) $a^i b = c^t$;

(iii) $a^{p+q}b = c^{r+s}$.

Case 1. If $t \geq r$ then from $(i), (ii)$ and Lemma 2.1, we obtain an arithmetic progression with a difference $i - p \leq q$, a contradiction.
Case 2. If $t < r$ then from $(ii), (iii)$ and Lemma 2.1, we obtain an arithmetic progression with a difference $p + q - i \leq q$, a contradiction. □

Definition 2.4. *An arithmetic progression* is a sequence of the form $a^k, a^{k+q}, a^{k+2q}, \cdots$ where $a \in A$, $k, q \in \mathbb{N}$, (so that the difference of any two consecutive powers is constant), and we call it a *minimal arithmetic progression* if q is the smallest possible number such that $a^k b = c^r$, $a^{k+q}b = c^{r+s}$ for some $k, r \in \mathbb{N}$, $s \in \mathbb{N} \cup 0$.

Definition 2.5. The *interval* on N_a of *length* L is the set

$$I = [a^t, a^{t+L}] = \{a^h \in N_a : t \leq h \leq t + L\}.$$

Lemma 2.6. *There exists* $P \in \mathbb{N}$ *such that the following holds. For every (not necessarily distinct)* $a, b \in A$, *and every* $x \in S$, *if* a^r *and* a^s $(r < s)$ *are the first two powers of* a *such that* $a^r x, a^s x \in N_b$ *then* $s - r \leq P$.

Proof. Consider x to be arbitrary but fixed. Within N_a there are at most $n = |A|$ minimal arithmetic progressions by Lemmas 2.1, 2.3, one for each N_b, $b \in A$. So we have sets A_{c_i} for $c_i \in \{c_1, c_2, \cdots, c_m\} \subseteq A$ where each A_{c_i} is the set of elements a^k such that $a^k x = c_i^r$, with the differences $d_1 \leq d_2 \leq \cdots \leq d_s \leq \cdots \leq d_m$ respectively. Thus, $N_a = H \cup A_{c_1} \cup A_{c_2} \cup \cdots \cup A_{c_s} \cup \cdots \cup A_{c_m}$ where $H = \{a, a^2, \ldots, a^p\}$

and $A_{c_s} = \{a^{p_s}, a^{p_s+d_s}, \dots\}$ such that $p_s = p + s$ for every $1 \leq s \leq m$ and then $A_{c_1} \cup A_{c_2} \cup \dots \cup A_{c_s} \cup \dots \cup A_{c_m}$ contains all but finitely many elements of N_a which is H by Lemma 2.1. Now, we prove that there exists $P \in \mathbb{N}$ not dependent on x, such that $d_s \leq P$ and this is sufficient since $a, b \in A$, A is finite and by taking the maximum of P over all a, b will do for all. Let us consider an interval I on N_a of length $L = d_1 d_2 \dots d_{s-1}$ which occurs at the point a^{p_M+1} where p_M is the maximum power among $\{p_1, p_2, \dots, p_s, \dots, p_m\}$.

Claim. I must contain at least one element from $A_{c_s} \cup A_{c_{s+1}} \cup \dots \cup A_{c_m}$.
PROOF. Suppose to the contrary that all the elements in I belong to $A_{c_1} \cup A_{c_2} \cup \dots \cup A_{c_{s-1}}$. Since A_i is an arithmetic progression with a difference d_i $(1 \leq i \leq s-1)$, and since $d_i | L$ it follows that $A_{c_{i+L}} \subseteq A_{c_i}$. Hence, if $I \subseteq A_{c_1} \cup A_{c_2} \cup \dots \cup A_{c_{s-1}}$, it follows that $I \cdot a^L \subseteq A_{c_1} \cup A_{c_2} \cup \dots \cup A_{c_{s-1}}$, and so $I \cdot a^{uL} \subseteq A_{c_1} \cup A_{c_2} \cup \dots \cup A_{c_{s-1}}$ for all $u \in \mathbb{N}$. Since I is an interval of length L, it follows that $\bigcup_{u \in \mathbb{N}} (I \cdot a^{uL})$ contains all but finitely many elements of $\mathbb{N}$. This contradicts the fact that A_{c_s} is an infinite set disjoint from all $A_{c_1}, A_{c_2}, \dots, A_{c_{s-1}}$. Therefore the claim has been proved. $\square$

Now we prove the lemma by induction on s. If $s = 1$ then we choose $L = 1$. Assume that the statement holds for every $k \leq s-1$. As a result of our claim, an interval J of length tL can be viewed as a disjoint union of t intervals of length L. Each of the latter contains a elements from $A_{c_s} \cup \dots \cup A_{c_m}$, and so J contains at least t such elements. Suppose that $d_s > L(m+1)$. So the interval $[a^r, a^{r+L(m+1)}]$ contains at least $m+1$ elements from $A_{c_s} \cup A_{c_{s+1}} \cup \dots \cup A_{c_m}$ and no elements from A_{c_s}. Then by using the pigeonhole principle, we conclude that two elements come from the same A_{c_t} $(s < t \leq m)$ with a difference less than d_s, a contradiction. Thus $d_s \leq L(m+1) \leq L(n+1)$. Since the number L is dependent on $d_1, \dots, d_{s-1}$, none of them is dependent on x by the induction hypothesis and by replacing m by n which is independent of x, we get $P = L(n+1)$ which does not depend on x. $\square$

The preceding result means that the differences of all minimal arithmetic progressions arising in Lemma 2.1 are uniformly bounded.

In the next lemma we prove that there is a uniform bound to how far arithmetic progressions can start.

Lemma 2.7. *There exists $Q \in \mathbb{N}$ such that the following holds. For every $a, b \in A$, and every $x \in S$, if a^r and a^s $(r < s)$ are the first two powers of a such that $a^r x, a^s x \in N_b$ then $r \leq Q$.*

Proof. Assume the opposite, i.e. that the start of an arithmetic progression can occur arbitrarily far into N_a, say beyond $T \geq (n+1)P$, where P is the constant in Lemma 2.6 and $n = |A|$. That means

$$a^T x = b^p, \;\; a^{T+d} x = b^q \tag{2}$$

Since the difference $d \leq P$ by Lemma 2.6, the n numbers $T - dk$ $(1 \leq k \leq n)$ are all positive. By using the pigeonhole principle, there are two distinct powers a^{T-hd}, a^{T-kd} where (without loss of generality) we assume $h > k$, such that

$a^{T-hd}x, a^{T-kd}x$ belong to the same N_c with a difference $(h-k)d$ where $b \neq c$ and that is clear because a^{T-hd}, a^{T-kd} appear before a^T, a^{T+d} where they are the first two powers such that (2) holds. Therefore, $a^{T+2(h-k)d}x \in N_c$ but this element also belongs to N_b where the power $T + 2(h-k)d \in \{T, T+d, \dots, T+md, \dots\}(m \in \mathbb{N})$, a contradiction. $\square$

Lemma 2.8. *As $x = b^s$ ranges over all of S, only finitely many arithmetic progressions arise in Lemma 2.1.*

Proof. Immediate consequence of Lemmas 2.6, 2.7 in which all these arithmetic progressions start within a bounded range and their periods are bounded as well. $\square$

3 Decidability for S

3.1 Word problem

A semigroup S generated by a finite set A has soluble word problem (with respect to A) if there exists an algorithm which, for any two words $u, v \in A^+$, decides whether the relation $u = v$ holds in S or not. For finitely generated semigroups it is easy to see that solubility of the word problem does not depend on the choice of (finite) generating set for S. We write $w_1 \equiv w_2$ if the words w_1 and w_2 are identical, and $w_1 = w_2$ if they represent the same element of S.

Remark 3.1. It is well known that finitely presented residually finite semigroups have soluble word problem [4], and from our results in [1], the semigroup under consideration in this paper is finitely presented and residually finite and then has soluble word problem. However, we give a concrete algorithm which is much more efficient than the generic one in the following theorem.

Theorem 3.2. *Every semigroup which is a disjoint union of finitely many copies of the free monogenic semigroup has soluble word problem.*

Proof. Let $S = \bigcup_{a \in A} N_a$, and $N_a = \langle a \rangle$. Thus the **Algorithm** is as follows:
Input: $u, v \in A^+$ and $u \equiv x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m}$ and $v \equiv y_1^{j_1} y_2^{j_2} \cdots y_n^{j_n}$, where $x_k, y_l \in A$ for every $1 \leq k \leq m$ and $1 \leq l \leq n$.

Output: $u = v$ or $u \neq v$ in S.

The idea of the **Algorithm** is to reduce the word u to an equivalent word a^k for some $a \in A$ and $k \in \mathbb{N}$.

Step 1.

Lemma 3.3. *If we have a presentation on S of the form*

$$\langle A |\ a^k b = [\alpha(a,k,b,1)]^{\kappa(a,k,b,1)},\ (a, b \in A,\ k \in \{1, 2, \dots, j\}) \rangle, \tag{3}$$

for some $\alpha(a,k,b,1) \in A$ and $\kappa(a,k,b,1) \in \mathbb{N}$, then there is an algorithm that reduces any word of the form a^ib to a word of the form c^j where $a,b,c \in A$ and $j \in \mathbb{N}$.

Proof. We specify the presentation (3) as follows. Firstly, notice that the relations in the presentation are of the form $x^iy = z^j$ where $x,y,z \in A$ and $i,j \in \mathbb{N}$ and thus we have at most $n(n-1)$ minimal arithmetic progressions in which we get at most $n(n-1)$ differences. Take the the least common multiple (LCM) of all these differences D. Thus

$$R'_{a,b} = \bigcup \{a^ib = [\alpha(a,i,b,1)]^{\kappa(a,i,b,1)} \ : i = 1,\dots,r(a,b)\},$$

Where $r(a,b) \leq Q = (n+1)P$ from 2.6, 2.7.

$$R_{a,b} = \bigcup_{k=r(a,b)+1}^{r(a,b)+D} \{a^kb = [\alpha(a,k,b,1)]^{\kappa(a,k,b,1)},\ a^{k+D}b = [\alpha(a,k+D,b,1)]^{\kappa(a,k+D,b,1)}\}, \tag{4}$$

and then we get the required presentation as

$$R = \bigcup_{a,b\in A} (R'_{a,b} \cup R_{a,b}),$$

where $k = lD$, l is any natural number. Notice that from (4) we have

$$a^{k+D}b = a^Da^kb = a^D[\alpha(a,k,b,1)]^{\kappa(a,k,b,1)} = [\alpha(a,k+D,b,1)]^{\kappa(a,k+D,b,1)} \tag{5}$$

So within N_a we have P_t arithmetic progressions, where t is the remainder of division of $r(a,b)+q$ by D for every $q \in \{1,2,\cdots,D\}$ as follows:

$$P_0 = \{a^{r(a,b)+1}, a^{r(a,b)+1+D}, a^{r(a,b)+1+2D}, \dots\},$$

$$P_1 = \{a^{r(a,b)+2}, a^{r(a,b)+2+D}, a^{r(a,b)+2+2D}, \dots\},$$

$$\vdots$$

$$P_{D-1} = \{a^{r(a,b)+D}, a^{r(a,b)+2D}, a^{r(a,b)+3D}, \dots\}.$$

Thus for every $i \in \mathbb{N}$ every word of the form a^ib is reduced to a word of the form c^j. □

Step 2.

Lemma 3.4. *For $a,b \in A$ and $s \in \mathbb{N}$, a finite number of applications of relations in R transforms a^sb to $[\alpha(a,r(a,b)+t+1+fD,b,1)]^{\kappa(a,r(a,b)+t+1+fD,b,1)}$ where $\alpha(a,r(a,b)+t+1+fD,b,1) \in A$ and $\kappa(a,r(a,b)+t+1+fD,b,1) \in \mathbb{N}$.*

Proof. If the relation

$$a^s b = [\alpha(a, r(a,b) + t + 1 + fD, b, 1)]^{\kappa(a,r(a,b)+t+1+fD,b,1)}$$

belongs to R, we are done. Now, suppose that the given relation does not appear in R, that means $s > k$ where $k = lD$ for some l and then $s = hD + t$ where $0 \le t < D$ and thus $a^s \in P_t$. Notice that P_t starts with the two elements $a^{r(a,b)+(t+1)}$, $a^{r(a,b)+(t+1+D)}$ and by doing some calculations as follows:
First we know that

$$s = hD + t,$$

and

$$s - r(a,b) - t - 1 = hD + t - r(a,b) - t - 1 = fD$$

for some f. Thus,

$$hD = fD + r(a,b) + 1.$$

So,

$$s = r(a,b) + t + 1 + fD,$$

which means that a^s is in the f position. Hence,

$$\begin{aligned}
a^s b &= a^{fD+r(a,b)+t+1} b \\
&\equiv \underbrace{a^D a^D \cdots a^D}_{f \text{ times}} a^{r(a,b)+t+1} b \\
&= \underbrace{a^D a^D \cdots a^D}_{f \text{ times}} [\alpha(a, r(a,b) + t + 1, b, 1)]^{\kappa(a,r(a,b)+t+1,b,1)} && \text{(by (4))} \\
&= \underbrace{a^D \cdots a^D}_{(f-1) \text{ times}} [\alpha(a, r(a,b) + t + 1 + D, b, 1)]^{\kappa(a,r(a,b)+t+1+D,b,1)} && \text{(by (5))} \\
&\vdots \\
&= [\alpha(a, r(a,b) + t + 1 + fD, b, 1)]^{\kappa(a,r(a,b)+t+1+fD,b,1)}.
\end{aligned}$$

Therefore, we can obtain $[\alpha(a, r(a,b) + t + 1 + fD, b, 1)]^{\kappa(a,r(a,b)+t+1+fD,b,1)}$ in finitely many steps. □

Step 3. Transform u to its normal form as follows:

$$\begin{aligned}
u &\equiv x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m} \\
&\equiv (x_1^{i_1} x_2) x_2^{i_2 - 1} \cdots x_m^{i_m} \\
&= x_{i_{12}}^{i_{12}} x_2^{i_2 - 1} \cdots x_m^{i_m} && \text{(by Lemma 3.4)} \\
&\equiv (x_{i_{12}}^{i_{12}} x_2) x_2^{i_2 - 2} \cdots x_m^{i_m}.
\end{aligned}$$

So, by taking the first power $x_1^{i_1}$ with the next element x_2 and doing this i_2 steps, we get rid of $x_2^{i_2}$ and using the same process with all $x_3^{i_3}, x_4^{i_4}, \cdots, x_m^{i_m}$, we end up with $x_I^{I_M}$ after $i_2 + i_3 + \cdots + i_m$ steps. So we have $u = x_I^{I_M}$.

Step 4. Transform v to its normal form $x_J^{J_N}$ analogously to step 3 .

Step 5. If $I = J$ and $I_M = J_N$ then $u = v$, otherwise $u \neq v$.

Therefore, S has soluble word problem. $\square$

3.2 Subsemigroup membership problem

Let S be a finitely generated semigroup. We say that S has a soluble subsemigroup membership problem if there is an algorithm that takes as input a finite set $Y = \{y_1, y_2, \cdots, y_k\} \subseteq S$ and an element $x \in S$ and decides whether x is in the subsemigroup T generated by Y.

Now we introduce necessary well-known theorems about subsemigroups of the natural number semigroup $\mathbb{N}$. We will use these theorems to devise an algorithm to solve the subsemigroups membership problem for the semigroup under consideration.

Theorem 3.5 ([6], Theorem 1)**.** *Let S be a subsemigroup of $\mathbb{N}$, then*

i) There is $s \in \mathbb{N}$ such that for $n \geq s$, $n \in S$, or

ii) There is $n \in \mathbb{N}$, $n > 1$ such that n is a factor of all $s \in S$.

We prove this theorem as the proof itself leads us to Corollary 3.10.
PROOF. Assume that there exist $s_1, s_2, \ldots, s_m \in S$ such that the g.c.d of the collection $(s_1, s_2, \ldots, s_m)$ is 1. Let S' be the subsemigroup of $\mathbb{N}$ generated by $\{s_1, s_2, \ldots, s_m\}$, notice that $S' \subseteq S$. Let $s = 2s_1s_2\ldots s_m$ and for $b > s$, since the g.c.d of $(s_1, s_2, \ldots, s_m)$ is 1, we may find integers $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that $\alpha_1 s_1 + \cdots + \alpha_m s_m = b$.
Hence there exist integers q_i and r_i such that $\alpha_i = q_i s_1 \ldots s_{i-1}s_{i+1}\ldots s_m + r_i$ where $0 < r_i \leq s_1 \ldots s_{i-1}s_{i+1}\ldots s_m$ $(i = 2, 3, \ldots, m)$. Now put

$$\beta_1 = \alpha_1 + (q_2 + \cdots + q_m)s_2s_3\ldots s_m,\ \beta_i = r_i,\ (i = 2, 3, \ldots, m).$$

Thus $b = \beta_1 s_1 + \beta_2 s_2 + \cdots + \beta_m s_m$. Note that $\beta_i > 0$ for $i = 2, 3\ldots, m$. But since

$$\beta_2 s_2 + \cdots + \beta_m s_m = r_2 s_2 + \cdots + r_m s_m \leq 2s_1s_2\ldots s_m < b,$$

clearly $\beta_1 > 0$. $\square$

Thus there are two types of subsemigroups of $\mathbb{N}$. The first type contains all natural numbers greater than some fixed natural number, and will be called relatively prime subsemigroups of $\mathbb{N}$. The second type is a fixed integral multiple of a relatively prime subsemigroup.

Corollary 3.6. *Every subsemigroup of $\mathbb{N}$ is finitely generated.*

Remark 3.7. This corollary is well known and here is an easy proof.

Proof. Suppose that S is a subsemigroup of $\mathbb{N}$ and the greatest common divisor of S is 1. Thus the generating set for S is $S \cap \{1, 2, \ldots, 2k\}$ where $k \in \mathbb{N}$ such that for every $n \geq k \ : \ n \in S$. Indeed this is so because if $m > 2k$ then $m = qk + f$. Thus $m = (q-1)k + k + f$ where $k + f \in S \cap \{1, 2, \ldots, 2k\}$. □

Fact: If S is a subsemigroup of $\mathbb{N}$ then the greatest common divisor g.c.d of S is the g.c.d of the generator set of S.

Corollary 3.8. *Every subsemigroup of* $\mathbb{N}$ *has the form*

$$F \cup D_{\mathcal{N},d},$$

where F is a finite set and $D_{\mathcal{N},d} = \{da \ : \ a \geq \mathcal{N}\}$.

Definition 3.9. Suppose that the semigroup S is generated by $\{n_1, n_2, \cdots, n_k\}$. If there exist two elements $d, \mathcal{N} \in S$ and a set $F \subseteq S$ such that

$$F = S \cap \{1, 2, \cdots, \mathcal{N} - 1\};$$

$$S \cap \{\mathcal{N}, \mathcal{N} + 1, \cdots\} = \{dk \ : \ k \in \mathbb{N}, \ dk \geq \mathcal{N}\},$$

then we say that S is defined by the *triple* $[d, \mathcal{N}, F]$.

Corollary 3.10. *Suppose that S is a subsemigroup of the natural number semigroup* $\mathbb{N}$. *Suppose that S is generated by* $n_1, n_2, \cdots, n_k$. *Then S is defined by the triple* $[d, \mathcal{N}, F]$ *where d is the greatest common divisor of* $\{n_1, n_2, \cdots, n_k\}$,

$$\mathcal{N} = 2dn_1 n_2 \cdots n_k,$$

and

$$F \subseteq \{1, 2, \cdots, \mathcal{N} - 1\}.$$

Proof. Follows immediately from Theorem 3.5 and Corollary 3.6. □

Corollary 3.11. *Suppose that S is a subsemigroup of the free monogenic semigroup N. Suppose that S is generated by* $a^{n_1}, a^{n_2}, \cdots, a^{n_k}$. *Then S is defined by the triple* $[d, \mathcal{N}, F]$ *where d is the greatest common divisor of* $\{a^{n_1}, a^{n_2}, \cdots, a^{n_k}\}$,

$$\mathcal{N} = a^2 d a^{n_1} a^{n_2} \cdots a^{n_k},$$

and

$$F \subseteq \{a, a^2, \cdots, a^{\mathcal{N}-1}\}.$$

Proof. Directly by Corollary 3.10. □

After understanding how subsemigroups of $\mathbb{N}$ behave we are ready to start designing the algorithm. Since

$$S = N_1 \cup N_2 \cup \cdots \cup N_n,$$

and T is a subsemigroup of S, then

$$T = T_1 \cup T_2 \cup \cdots \cup T_m,$$

where $T_i \leq N_i$ for every $i \in \{1, 2, \cdots, m\}$, $m \leq n$. Consequently, the generator set for T is

$$A_T = \bigcup_{i \in \{1,2,\cdots,m\}} A_{T_i},$$

where A_{T_i} is the generator set of T_i for every $i \in \{1, 2, \cdots, m\}$. Thus T is finitely generated ([3], Proposition 3.1).

Lemma 3.12. *Suppose that the subsemigroup $U_j = \langle N_j \cap A_T \rangle$ is defined by the triple $[d_j, \mathcal{N}_j, F_j]$. Then there is an algorithm which takes arbitrary U_i, U_j and $b \in A_T$ and tests whether*

$$U_i b \cap N_j \subseteq U_j$$

or not.

Proof. Let $a_j^r \in U_i b \cap N_j$. Then

$$a_j^r \in U_j \iff a_j^r \in F_j \text{ or } a_j^r = a_j^{d_j h_j} \text{ for some } d_j h_j \geq d_j t_j \text{ where } d_j t_j = \mathcal{N}_j,$$

by Corollary 3.10. □

Theorem 3.13. *Every semigroup which is a disjoint union of finitely many copies of the free monogenic semigroup has a soluble subsemigroup membership problem.*

Proof. Let S be such a semigroup. Then the **Algorithm** is as follows:
Input. A finite set $A_T \subseteq S$, specified as normal form words over the generating set, and an element $x \in S$, specified as a normal form word.

Output. Whether $x \in T$ where T is the subsemigroup of S generated by A_T.

Step 1. Take $U_i = \langle A_i \rangle = \langle N_i \cap A_t \rangle$, which means that U_i is a finitely generated subsemigroup of N_i and then is defined by the triple $[d_i, \mathcal{N}_i, F_i]$ by Corollary 3.11. Now check if

$$(U_1, U_2, \ldots, U_m) = T$$

where

$$(U_1, U_2, \ldots, U_m) = U_1 \cup U_2 \cup \cdots \cup U_m,$$

which means check whether

$$U_i x \subseteq \bigcup_{i=1}^{m} U_i \text{ for every } i \in \{1, 2, \cdots, m\} \text{ and for every } x \in A_T,$$

by Lemma 3.12. If yes then go to step 4. If there was $a_i^{r_i} x = a_j^{r_j}$ and $a_j^{r_j} \notin U_j$ then go to step 2.

Step 2. Add the missing element $a_j^{r_j}$ to U_j and then we have

$$U_j^{(+1)} = \langle A_{U_j} \cup a_j^{r_j} \rangle, \text{ which is defind by the triple } [d_j^{(+1)}, \mathcal{N}_j^{(+1)}, F_j^{(+1)}].$$

Notice that by adding $a_j^{r_j}$ to U_j we reduce the gaps in F_j or we reduce the difference d_j by Corollary 3.8. Thus we get the new description

$$(U_1, U_2, \cdots, U_{j-1}, U_j^{(+1)}, U_{j+1}, \cdots, U_m), \tag{6}$$

Step 3. We start again with the new description (6) and we keep adding these missing elements with all $i \in \{1, 2, \cdots, m\}$. And then we reach to the final description

$$(U_1^{(+s_1)}, U_2^{(+s_2)}, \cdots, U_j^{(+s_j)}, \cdots, U_m^{(+s_m)}) = T.$$

Which means that $U_j^{(+s_j)} b \subseteq \bigcup\limits_{i=1}^{m} U_i^{(+s_i)}$ for every $b \in A_T$ and for every $j \in \{1, 2, \cdots, m\}$ and that because as we explained before each U_j is defined by the triple $[d_j, \mathcal{N}_j, F_j]$. So if we add an element $a_j^{r_j}$ to U_j that means, by Corollary 3.8, we reduce the gaps in F_j and they are finite, or we reduce the difference d_j and we can do this just finitely often. Thus we add finitely many elements in each U_j, which implies that this process terminates. So now each $U_j^{(+s_j)}$ is defined by the triple

$$[d_j^{(+s_j)}, \mathcal{N}_j^{(+s_j)}, F_j^{(+s_j)}].$$

Step 4. If we were given $x = a_h^{r_h} \in S$ and we want to see if $x \in T$ or not then we just take this element and see in $U_h^{(+s_h)}$ if

$$a_h^{r_h} \in F_h^{(+s_h)},$$

or

$$r_h = d_h^{(+s_h)} k \text{ for some } d_h^{(+s_h)} k \geq d_h^{(+s_h)} t \text{ where } d_h^{(+s_h)} t = \mathcal{N}_h^{(+s_h)},$$

then $x \in T$ otherwise $x \notin T$.

□

Acknowledgment

I would like to thank the reviewer for their detailed comments and constructive suggestions for the manuscript. The author wants to thank Prof Nik Ruškuc (University of St Andrews) for his many helpful contributions to this work. And to thank Prof Pavel Etingof (MIT) for his useful comments and suggestions on this paper.

References

[1] N. Abughazalah, N. Ruškuc, On disjoint unions of finitely many copies of the free monogenic semigroup, *Semigroup Forum*, **87** (2013), No. 1, Springer US, pp 243–256.

[2] N. Abughazalah, J. D Mitchell, Y. Péresse, N. Ruškuc, A classification of disjoint unions of two or three free monogenic semigroup, *Semigroup Forum*, **91** (2015), Issue 1, pp 53–61.

[3] I.M. Araújo, M.J.J. Branco, V.H. Fernandes, G.M.S. Gomes, and N. Ruškuc, On generators and relations for unions of semigroups, *Semigroup Forum*, **63** (2001), pp 49–62.

[4] E.A. Golubov. Finitely approximable regular semigroups (Russian), *Mat. Zametki*, **17** (1975), pp 423–432. (English translation: Math. Notes 17 (1975) 247–251.)

[5] P.A. Grillet, *Semigroups*, Marcel Dekker, New York, 1995.

[6] J. Higgins, Subsemigroups of the additive positive integers, *Fibonacci Quart*, **10** (1972), 225–230.

[7] J.M. Howie, *Fundamentals of Semigroup Theory*, Clarendon Press, Oxford, 1995.